Algebraic Metacomplexity and Representation Theory

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with

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Intro: algebraic circuits

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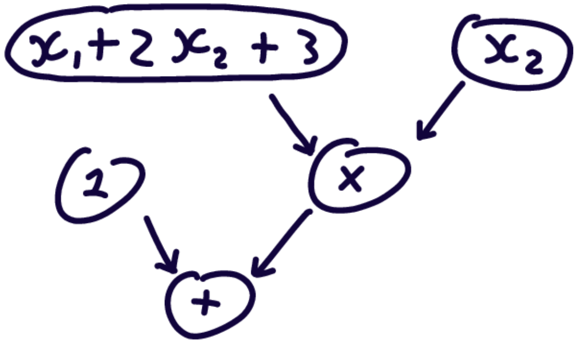

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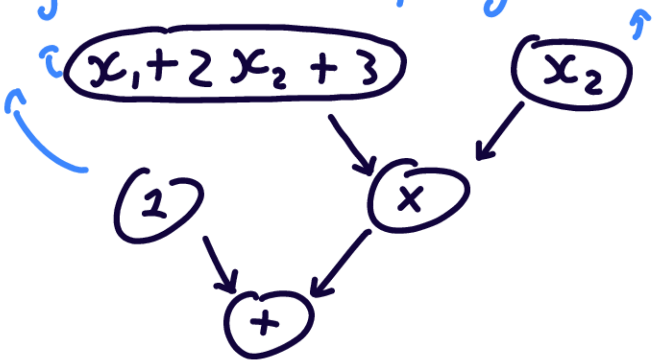
Intro: algebraic circuits

input gates : linear polynomials



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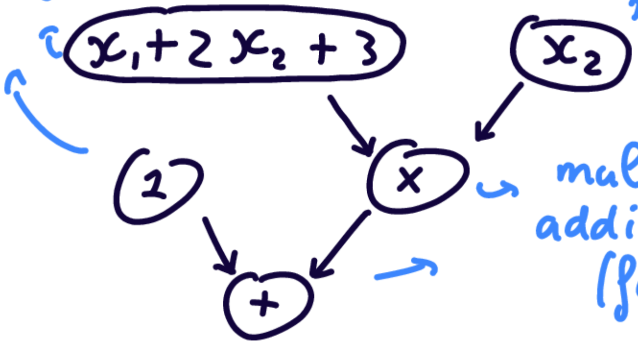

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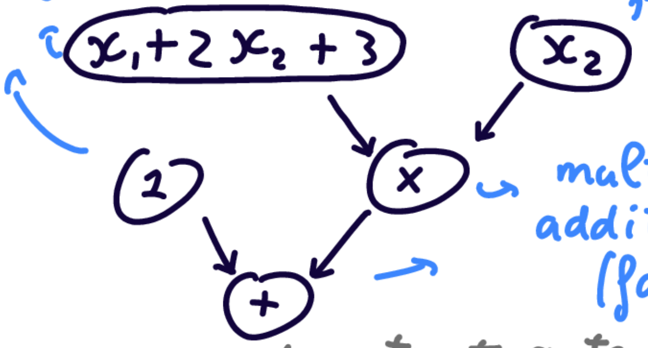

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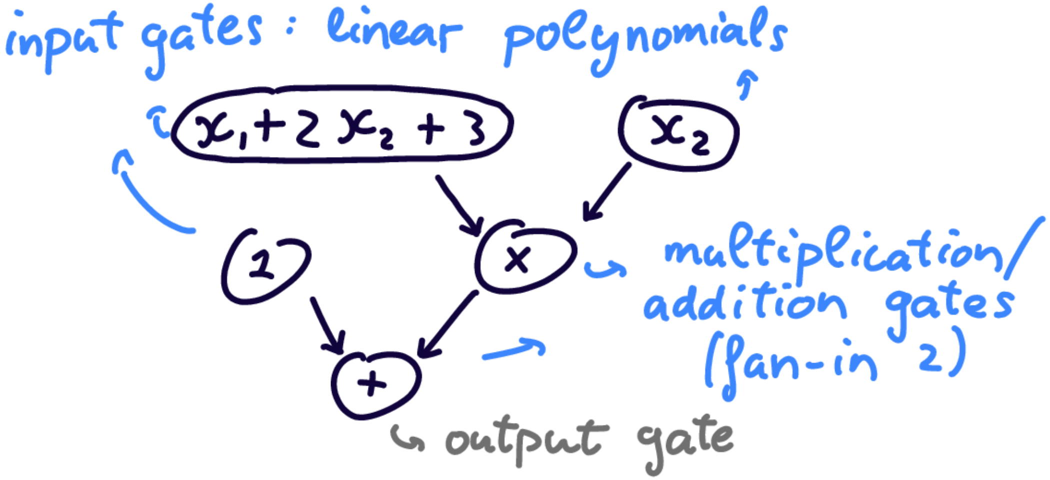

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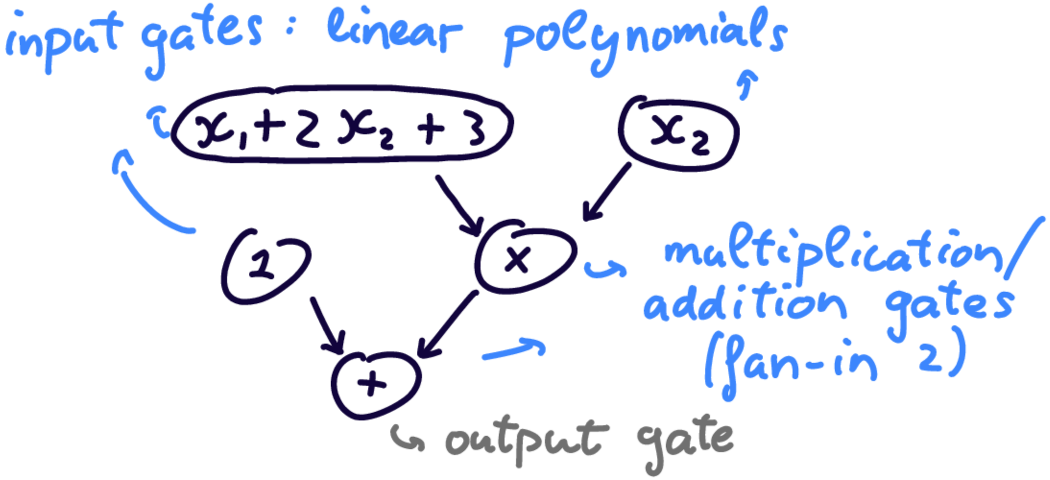

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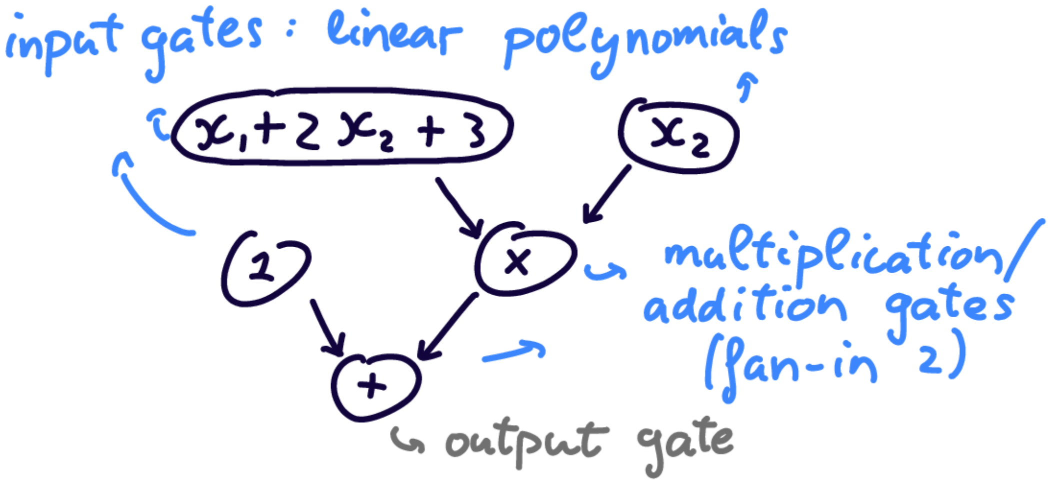

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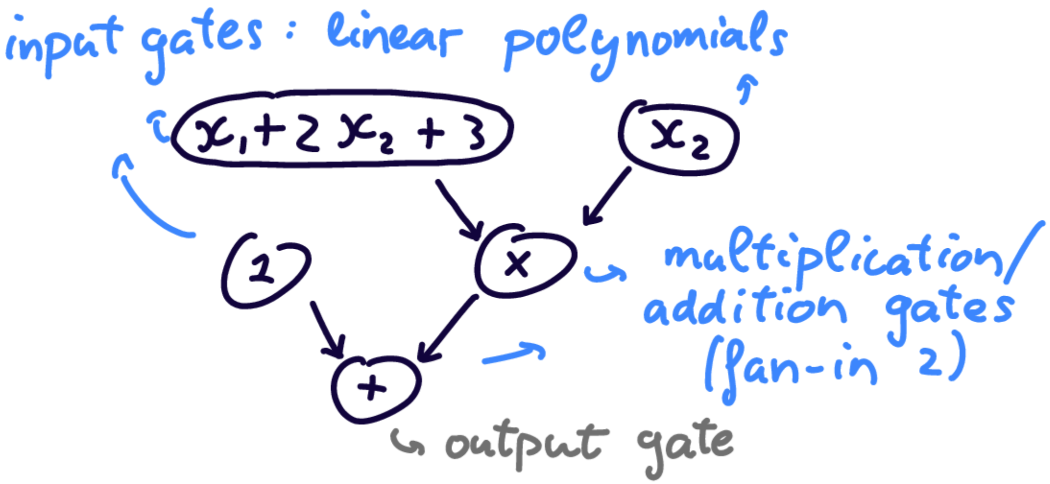

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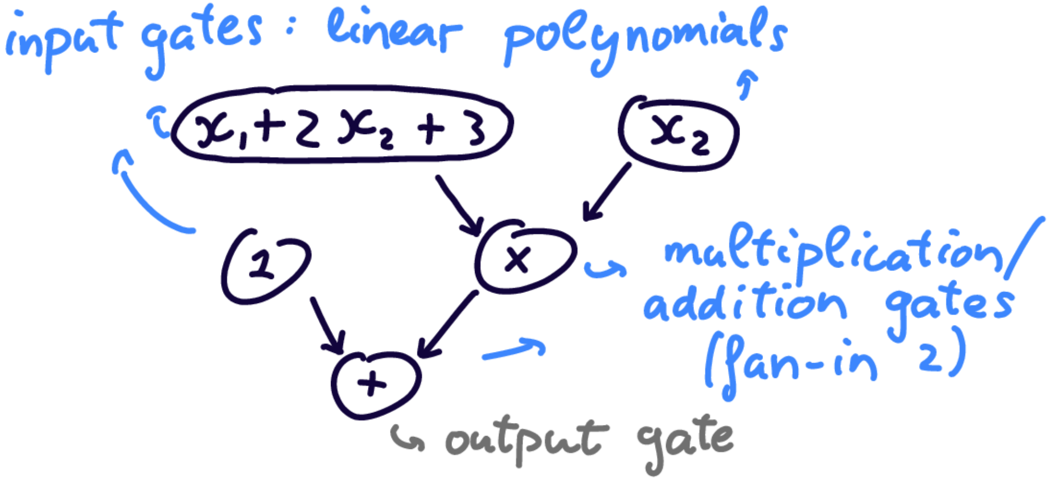

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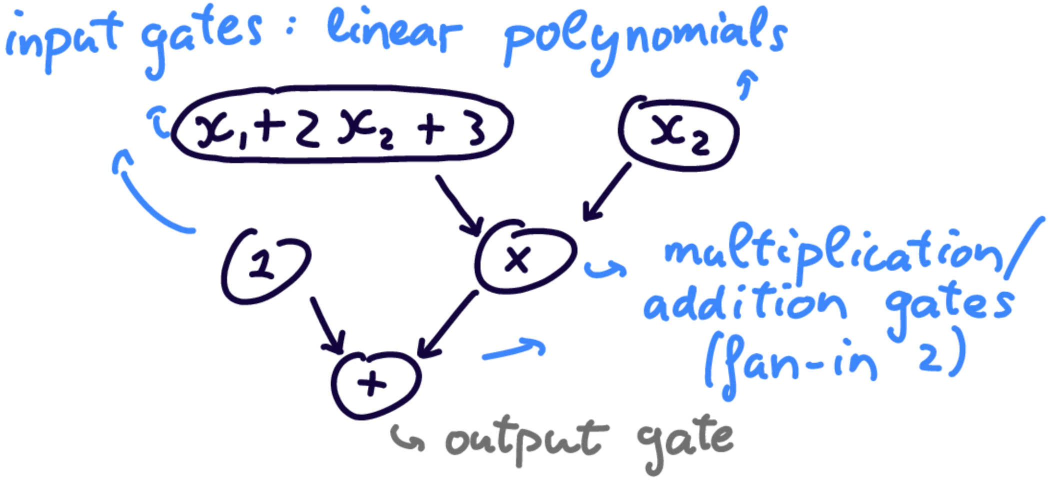

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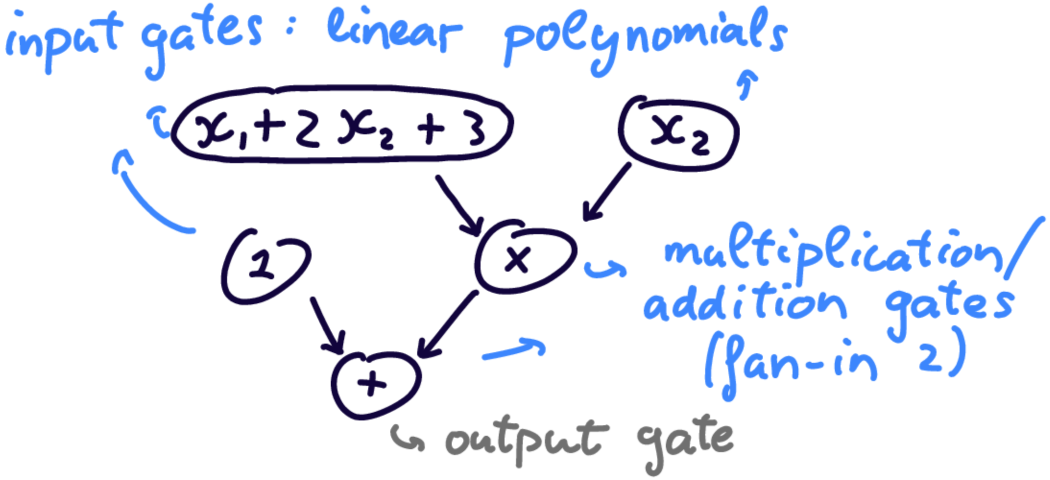

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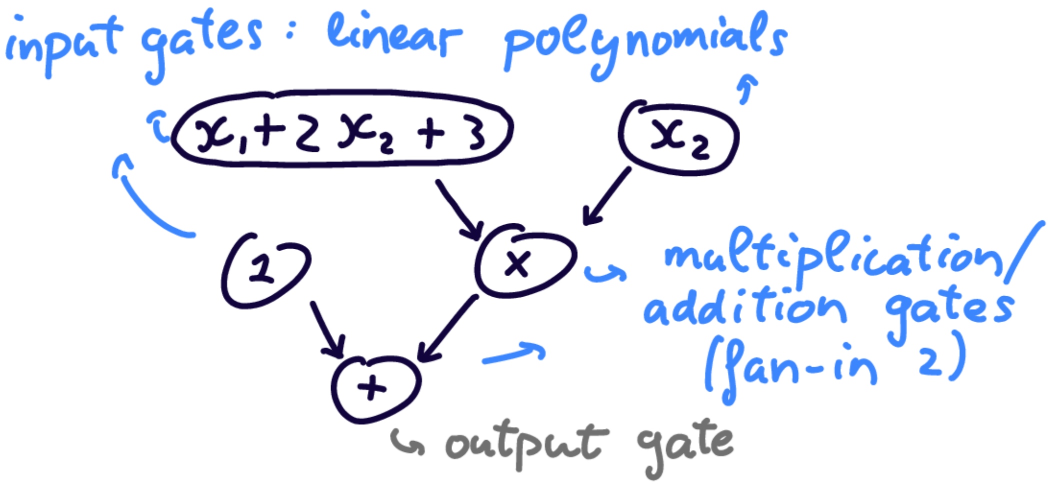

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↳ allows replacing cc with other complexity measures (algebraic branching programs, determinental complexity, etc.)

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The goal: find analogs in the algebraic world

- rank method barriers [Efremenko-Garg-Oliveira-Wigderson'18]
- ongoing work: is there an **algebraic natural proofs barrier**?

  [Grochow-Kumar-Saks-Saraf'17] [Forbes-Shpilka-Volk'18]
  [Chatterjee-Kumar-Ramya-Saptharishi-Tengse'20] [Kumar-Ramya-Saptharishi-Tengse'22]

Geometric complexity theory

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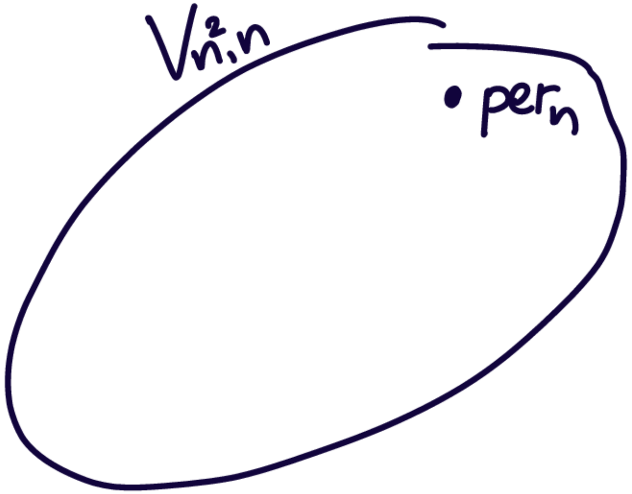

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$V_{k,n} := \mathbb{C}[x_1, \dots, x_k]_n$ homogeneous pols of degree n

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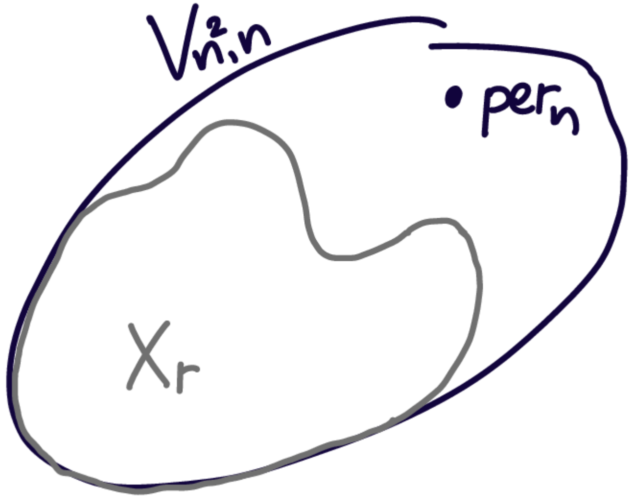

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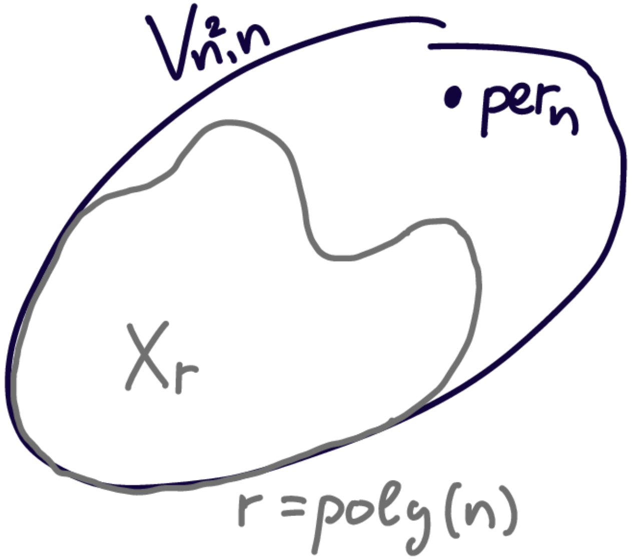

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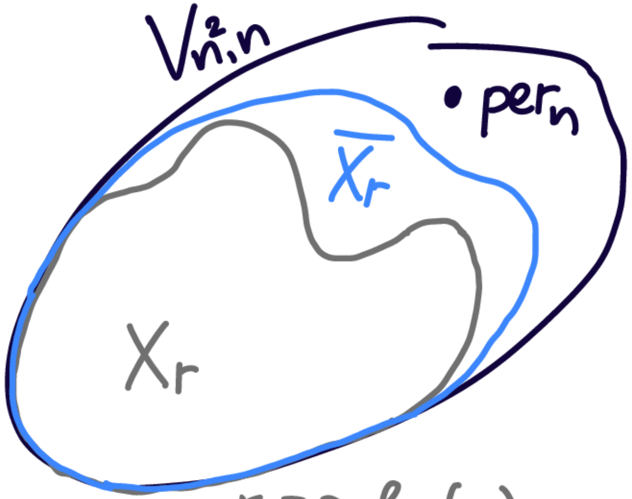

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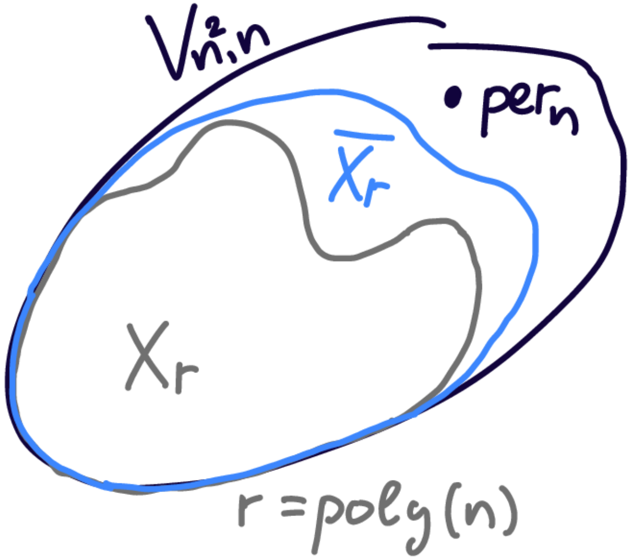

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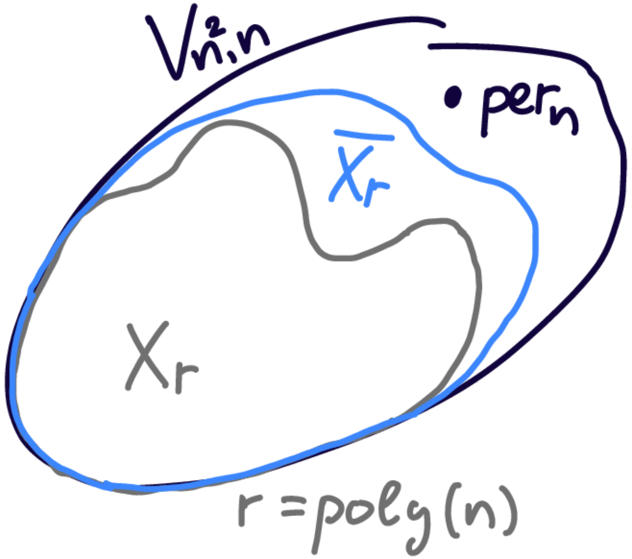

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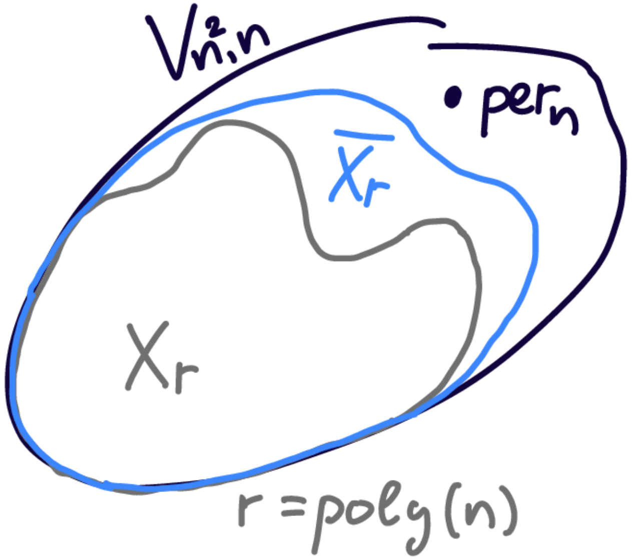

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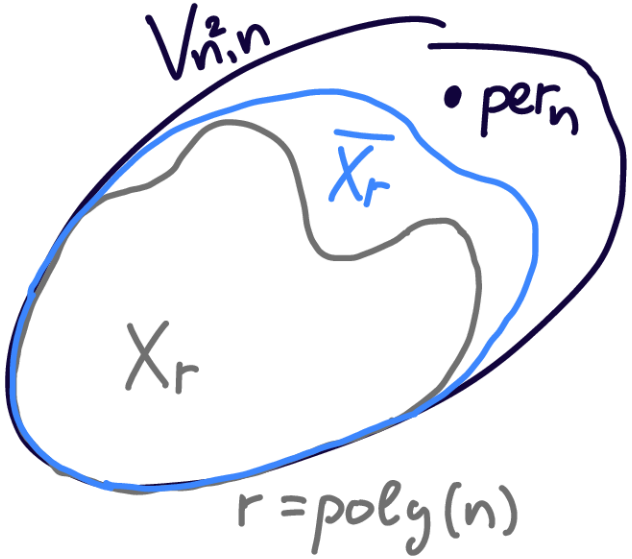

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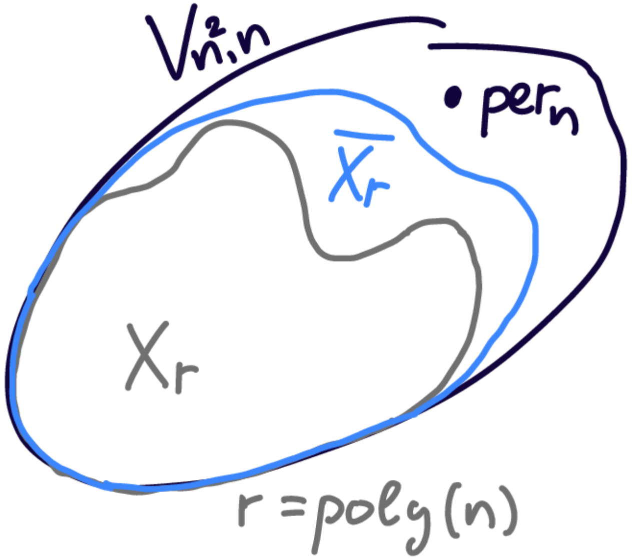

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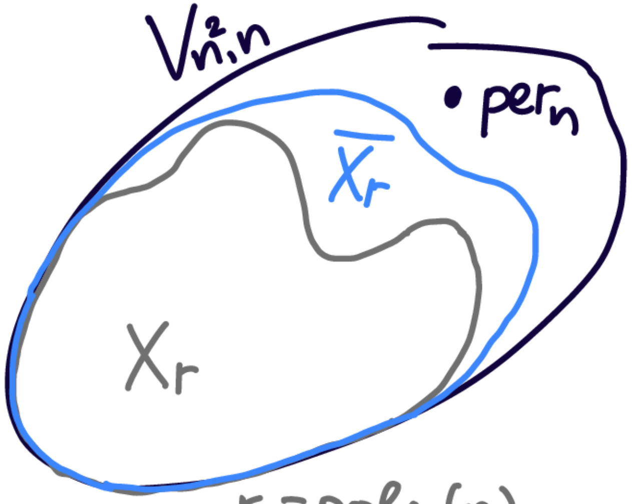

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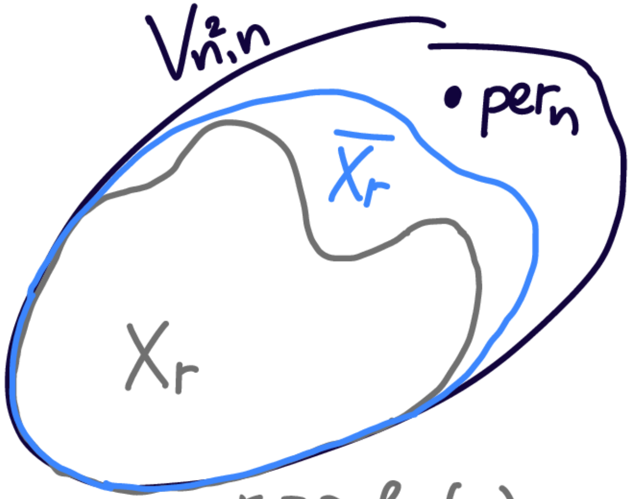

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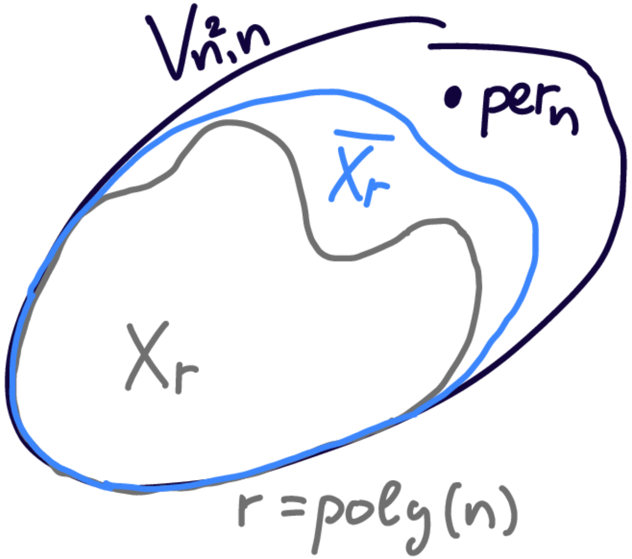

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$$\Delta(\overline{X_r}) = \{0\}$$

$$\Delta(per_n) \neq 0$$

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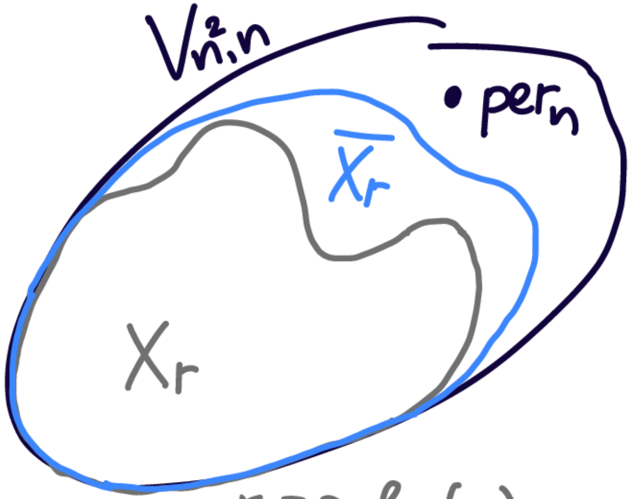

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now vary n:

- $X_{r(n)}$: "low complexity pols"
  $r(n) = poly(n)$
- $\Delta^{(n)}$: "separating metapolynomials" →

- $\Delta^{(n)}(\overline{X_{r(n)}}) = \{0\}$
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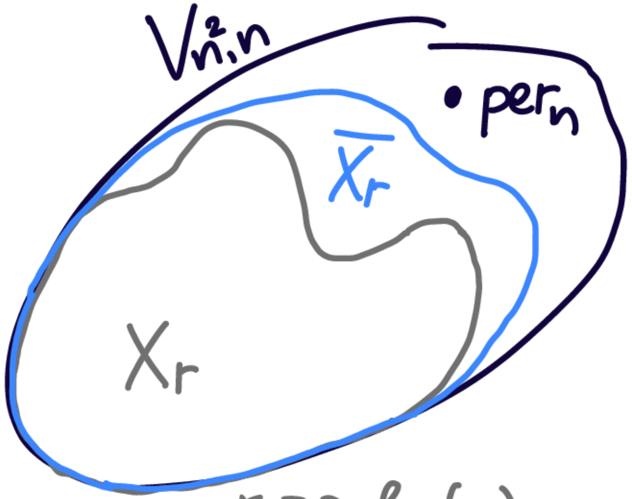

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- $\Delta^{(n)}$: "separating metapolynomials"

$\boxed{\begin{array}{l} \textbf{Conjecture: } \text{for infinitely many } n, \\ \bullet\ \Delta^{(n)}(\overline{X_{r(n)}}) = \{0\} \\ \bullet\ \Delta^{(n)}(per_n) \neq 0 \end{array}}$

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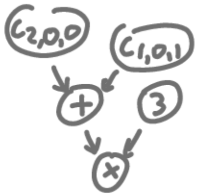

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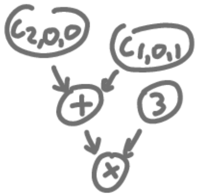

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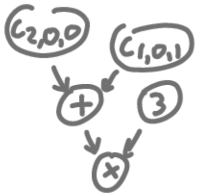

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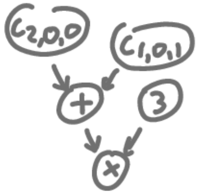

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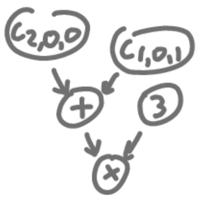

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Our result: sufficient to prove this for metapols with strong representation-theoretic properties

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superpolynomial cc lower bounds for **homogeneous** $\Delta^{(n)}$
$\Rightarrow$ **general** superpolynomial lower bounds

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$V_{k,n} := \mathbb{C}[x_1, \dots, x_k]_n$

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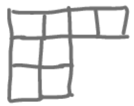

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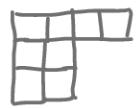

- grouping by type: $\mathbb{C}[V_{k,n}] = \bigoplus_{\delta \geq 0} \bigoplus_{\lambda \vdash \delta n} \mathbb{C}[V_{k,n}]_\lambda$ isotypic subspace

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&+ \Big[\, 2\big(-c_{020}c_{101}^2 - 2c_{011}c_{101}c_{110} + 4c_{002}c_{110}^2 - c_{011}^2 c_{200} + 8c_{002}c_{020}c_{200}\big) \\
&+ \quad 5\big(c_{020}c_{101}^2 - c_{011}c_{101}c_{110} - c_{002}c_{110}^2 + c_{011}^2 c_{200} + 4c_{002}c_{020}c_{200}\big) \Big] \\
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$$\mathbb{C}[V_{k,n}]_\lambda \cong V_\lambda^{\oplus p_{k,n}^\lambda}$$

↳ irreducible representation

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\end{aligned}$$

$$\mathbb{C}[V_{k,n}]_\lambda \cong V_\lambda^{\oplus p_{k,n}^\lambda}$$

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We know bases for V_λ:

Representation theory

$$\bigoplus_{\delta \geq 0} \mathbb{C}[V_{k,n}]_\delta = \bigoplus_{\delta \geq 0} \bigoplus_{\lambda \vdash \delta n} \mathbb{C}[V_{k,n}]_\lambda$$

$$\Delta = c_{200} c_{020} c_{002} = \Delta_{\ydiagram{6}} + \Delta_{\ydiagram{4,2}} + \Delta_{\ydiagram{2,2,2}}$$

$k=3 \quad n=2 \quad \delta = 3$

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60\,\Delta &= 2\left(c_{101}^2 c_{020} + 2c_{110}c_{101}c_{011} + c_{200}c_{011}^2 + c_{110}^2 c_{002} + 2c_{200}c_{020}c_{002}\right) \\
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Why rep. theory?

$$\boxed{\begin{aligned}\Delta^{(n)}\left(X_{r(n)}\right) &= \{0\} \\ \Delta^{(n)}\left(\rho_n^{\text{hard}}\right) &\neq 0\end{aligned}}$$

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and similar for weight decompositions & highest weight vectors

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$\longrightarrow$ reducible via general Lie algebra relations

$\hookrightarrow$ the Poincaré-Birkhoff-Witt theorem

Thanks!